FRAMES OF REFERENCE AND SOME OF ITS APPLICATIONS

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Abstract

We define a Frame of reference as a two ingredients concept: A meta-rigid motion, which is a generalization of a Born motion, and a chorodesic synchronization, which is an adapted foliation. At the end of the line we uncover a low-level 3-dimensional geometry with constant curvature and a corresponding coordinated proper-time scale. We discuss all these aspects both from the geometrical point of view as from the point of view of some of the physical applications derived from them.

1 Introduction

Choosing a system of units is the first decision that a physicist has to take to communicate the outcome of an experiment or the quantized prediction of a theory. Metrology is therefore in some sense the more universal branch of physics.

Next in importance is to remind the frame of reference which has been used to set up the experiment if it has already been done or to describe the conditions under which the experiment has to be set.

The systems of units have been chosen with the requirement that they can be conveniently reproduced, or identified, to the desired accuracy by anybody who has at his disposal the necessary equipment. A family of allowed frames of reference has to be chosen keeping in mind the same requirement but many

other specific ones are necessary to cope with the increased complexity of the concept. And more important, not only is necessary to describe the hardware and the protocols involved but also the theory which is supposed to describe the concept.

Switching from a system of units to another is a very easy task. Comparing data referred to different frames of reference may be a very difficult one, depending on the domain which is being considered and the theory which is available.

The concepts of *Absolute time* and *Absolute space* very much simplify the theory of frames of reference in classical physics. This theory relies heavily on the concept of *Rigid motions*, and the behavior of solids as as objects which are able to move rigidly, or approximately so at a desired level of accuracy, under appropriate conditions.

Special and General relativity do not have a theory of Frames of reference as complete and satisfactory as Classical physics. In Special relativity only the theory of Galilean frames of reference or that of irrotational Born motions is firmly established. In General relativity only those space-times possessing symmetries or Born congruences can be said to have frames of reference everybody will agree upon. But in any case the number of such particular frames of reference is never comparable to the number offered by Classical physics.

There are two end of the line ingredients in the concept of a frame of reference: i) A time scale and ii) A low-level of geometry, i.e. the geometry of the metrology.

As Helmholtz [1], Poincaré [2] and Cartan [3] stressed, this geometry has to possess the property of free mobility, meaning that those objects that will be used as standards for measuring must remain invariant when they move around. This is a very restrictive property which requires this low-level geometry to be a Riemannian geometry with constant curvature.

Of course the word geometry has different meanings depending on the context on which it is used. The fact that in Relativistic physics one has to do more often with space geometries which do not have constant Riemannian curvature is by no means a contradiction. These geometries are an essential part of the mathematical formalism and they should be interpreted appropriately, but not necessarily as low-level geometries. On the other hand the low-level geometry may be deeply hidden in the formalism and has to be uncovered.

Going from Absolute time of classical mechanics to a relativistic time-scale based on proper-time coordination is a very innovative aspect of Relativity physics but does not present any real conceptual or mathematical difficulty. This is not so when abandoning the concept of Absolute space the concept of the relativity of space is considered with its implications in the process of measuring lengths, surfaces or volumes. The culprit lies in the difficulty to define rigid motions. The generalization of this concept is the first problem that a consistent theory of the frames of reference has to face.

This contribution is an update of early papers dealing with the subject of frames of reference in Special and General relativity written since 1990. It contains very little new material. The aim it is more to look at some of the applications emphasizing what they have in common and showing why a theory of frames of reference is necessary to give a meaning to the problems these applications deal with, or to produce numerical results everybody can understand.

The compilation that we present here includes an introduction to the theory of Frames of reference in Relativity physics and the presentation of five scaled down versions of the applications, already published or posted, that this theory motivated. We hope that understanding these applications will convince the reader of the necessity of developing such a theory.

Section 2 gives a rather detailed description of the theory of the frames of reference introducing the concepts of quo-harmonic congruences, meta-rigid motions, principal transformations, chorodesic synchronizations and atomic time scale coordination, to which we shall refer as *Temps Atomique Coordonné (TAC)*. A previous reading of this section is necessary to understand the vocabulary and some of the fine points made in sections 2-7. But each of these sections are independent one from the other.

Section 3 deals with the concept of strain from the unavoidable point of view that requires that to define the strain of a body it is necessary to compare it to another object, possibly an idealized one, which serves as reference in the sense that it is harder to strain than the object being tested.

Section 4 is devoted to the analysis of the anisotropy of space in a frame of reference co-moving with the Earth. It is an important application because it leads to a prediction about the outcome of experiments of the Michelson-Morley type when sensitivities of the order of $10^{-12} - 10^{-13}$ can be reached.

Section 5 introduces the concept of *Modes* of a quantized scalar field in Milne's universe, [23]. Although it is not more difficult to deal with a general

Robertson-Walker model we have preferred to use this very simple model because besides the TAC another important aspect plays a role in this case. Namely the necessity of taking into account global properties of space-times when selecting the family of allowed quo-harmonic congruences which are acceptable as meta-rigid motions of frames of reference.

Section 6 tackles the problem of identifying the mathematical meaning of the family of transformations between two systems of quo-harmonic coordinates adapted to two different frames of reference. We consider as a simple example the family of Born congruences in Minkowski's space-time.

Section 7 is an essay at justifying a very simple cosmological model which is based on an equation of state for a gas of zero mass particles which differs from the usual one. Our equation of state is deeply anchored in the theory of frames of reference of section 2.

The Appendix is a short summary of definitions and notations concerning the geometry of time-like congruences. The reader which is not familiar with the subject is advised to start his reading with this appendix.

In all sections but the last (sect. 7) the system of units is supposed to be such that the universal constant c is equal to 1.

2 Frames of Reference

We give below a precise and intrinsic definition of what we mean by a *Frame of reference*, thus completing tentative definitions presented in previous publications [4], [5], [6], [7], [8].

A frame of reference is a pair of geometrical objects intrinsically defined in the space-time being considered:

- A time-like congruence $\mathcal{R}$ of a particular type, that we shall call the *Meta-rigid motion* of the frame of reference, with its corresponding *Principal transformation* which allows to assign a constant number to each pair of world-lines, to be interpreted as its physical distance, and to implement the free mobility of ideal rigid bodies, thus uncovering a low-level geometry.

- A space-like foliation of a particular type $\mathcal{F}$, that we shall call the *Chorodesic synchronization* of the frame of reference, with its corresponding (*Atomic time scale coordination*).

2.1 *Meta-rigid motions*

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In classical physics rigid motions are defined as those congruences which keep constant the euclidean distance between any two simultaneous events with respect to absolute time. That is to say:

$$t = t, \quad x^i(t) = R^i_j(t)\left[z^j - A^j(t)\right], \tag{1}$$

where $x^i(t)$ and z^i are cartesian coordinates and $R^i_j(t)$ are rotation matrices. Two main properties of these congruences are worth mentioning here besides some other obvious ones:

i) Let $v^i(t, z^j)$ be the velocity field of any congruence. Then rigid congruences can be characterized by the local condition:

$$\sigma_{ij} = \partial_i v_j + \partial_j v_i = 0 \tag{2}$$

ii) the knowledge of $a^j(t) = \dot{v}^j$, where the dot means a derivative with respect to time, and the rotation rate $\omega_{ij} = \partial_i v_j - \partial_j v_i$ along one of the world-lines defines the whole congruence.

iii) The rigid-motion congruences are homogeneous in the sense that none of the world-lines is privileged in the definition, and every one together with their corresponding rotation rate field may be considered to be the seed of the same congruence.

To define the type of admissible meta-rigid motions $\mathcal{R}$ of relativistic physics three general conditions have to be imposed which are meant to make this concept to inherit as much as possible of the formal properties of rigid motions in classical physics:

i) Their local characterization must be intrinsic and have a meaning independently of the space-time being considered. But boundary and global conditions must be appropriate to each particular case.

ii) The knowledge of any open sub-bundle of the congruence must be sufficient to characterize the whole congruence.

iii) The congruence has to be homogeneous, i.e. its definition does not have to distinguish any particular world-line of the congruence.

To comply with these three conditions the most natural approach is to require the vector field u^α of the meta-rigid motions to satisfy a set of differ-

ential conditions. Obvious candidates which satisfy this requirement are the differential equations defining the Killing congruences.

$$\Sigma_{ij} = \hat{\partial}_t \hat{g}_{ij} = 0, \quad \partial_i \Lambda_j - \partial_j \Lambda_i = 0. \tag{3}$$

Other congruences to be acceptable as meta-rigid motions are the Born congruences, which are a generalization of the Killing congruences and are defined by the single group of conditions:

$$\partial_t \hat{g}_{ij} = 0. \tag{4}$$

We shall continue to call these congruences Rigid congruences, as usual. Meta-rigid ones will be an appropriate generalization of them.

The Born conditions, and a fortiori the Killing conditions, are very restrictive in any space-time including Minkowski's one, [9], [10], and the question may be raised of generalizing the concept of meta-rigid motion in such a way that the family of congruences $\mathcal{R}$ generalize Born congruences up to the point of having hopefully a number of degrees of freedom equivalent to that of the group of rigid motions in classical mechanics [1].

When facing this problem, usually forced by a physical application which requires the sublimated modelisation of the behavior of a rigid body, many authors recur to use Fermi congruences in its general form, [11]. By this we mean the time-like congruences which world-lines have constant Fermi coordinates based on a distinguished world-line seed. Because of this distinction, except in the case where they are in fact Born congruences, Fermi congruences are not acceptable candidates to the concept of a meta-rigid motion [14].

Harmonic congruences can be defined as those time-like congruences u^α for which there exist three space-like functions $f^a(x^\alpha)$ satisfying the following equations:

$$\Box f^a = 0, \quad u^\alpha \partial_\alpha f^a = 0, \quad a, b, c = 1, 2, 3 \tag{5}$$

where $\Box$ is the d'Alembertian operator corresponding to the space-time metric.

[1] In references [7] and [8] some other apparently natural generalizations were suggested along this line but they all looked too restrictive in three dimensions of space

Harmonic congruences, with the corresponding harmonic coordinates, have been used extensively in the literature for several reasons, including mainly the fact that they simplify many calculations, even if now and then some inconsistencies have been pointed out in some particular problems [12]. Harmonic congruences could in principle be considered good representatives of meta-rigid motions. In fact they are not. They are acceptable generalizations of Killing congruences but they are not generalizations of the Born congruences. More precisely it has been proved [13] that irrotational Born congruences, which are not Killing, are never Harmonic congruences. A familiar example of such congruence is the Born congruence generated by an arbitrary non geodesic world-line of Minkowski space-time.

To solve the problem just mentioned concerning the harmonic congruences it has been proposed to identify the meta-rigid motions with appropriately selected *Quo-harmonic congruences*, [14], [7], [8]. The latter being those congruences for which there exist three independent space-like solutions $f^a(x^\alpha)$ satisfying the following equations:

$$\hat{\triangle} f^a = 0 \tag{6}$$

where:

$$\hat{\triangle} \equiv \hat{g}^{ij}(\hat{\partial}_i\hat{\partial}_j - \hat{\Gamma}^k_{ij}\hat{\partial}_k) \tag{7}$$

and where $\hat{\Gamma}^k_{ij}$ are the symbols defined in 123. These are proper generalizations of Born congruences in the sense that any Born congruence is also a quo-harmonic one because in this case $\hat{g}_{ij}(x^k)$ does not depend on time and the operator $\hat{\triangle}$ is the usual Laplacian operator corresponding to a 3-dimensional Riemannian metric. It has been impossible up to now to determine, or even to estimate, the number of degrees of freedom of the class of quo-harmonic congruences in a general space-time except that it is certainly more general than that of Born congruences and it even looks too general, letting us to think that global conditions, boundary conditions, or supplementary physical conditions, should be invoked to restrict this generality.

Notice that if a congruence is quo-harmonic and quo-harmonic coordinates are used, $x^i = f^i$, then:

$$\hat{g}^{ij}\hat{\Gamma}^k_{ij} = 0 \tag{8}$$

Obviously, if adapted space coordinates of a congruence can be found such that the preceding conditions are satisfied then this congruence is quo-harmonic.

2.2 *Principal transformations of the Fermat quo-tensor*

Let us consider the Zel'manov-Cattaneo tensor 124. We shall say that a skew-symmetric quo-tensor $\hat{f}^{ij}$ is an eigen quo-form if it gives a stationary value to the function:

$$\sigma \equiv -\frac{\hat{R}_{ijkl}\hat{f}^{ij}\hat{f}^{kl}}{\hat{g}_{ijkl}\hat{f}^{ij}\hat{f}^{kl}} \tag{9}$$

where:

$$\hat{R}_{ijkl} = \hat{g}_{is}\hat{R}^{s}_{jkl}, \quad \hat{g}_{ijkl} = \hat{g}_{ik}\hat{g}_{jl} - \hat{g}_{jk}\hat{g}_{il} \tag{10}$$

which will be by definition the corresponding eigen-value. Equivalently we can consider this other tensor, [15] and [4]:

$$\tilde{R}_{ijkl} = \frac{1}{4}(\hat{R}_{ijkl} - \hat{R}_{jikl} + \hat{R}_{klij} - \hat{R}_{lkij}) \tag{11}$$

Since this quo-tensor has the symmetries of a Riemann tensor in three dimensions we have the following well known identity, which expresses that the Weyl-like object is zero:

$$\tilde{R}_{ijkl} = \hat{g}_{ik}\tilde{R}_{jl} - \hat{g}_{il}\tilde{R}_{jk} + \hat{g}_{jl}\tilde{R}_{ik} - \hat{g}_{jk}\tilde{R}_{il} - \frac{1}{2}R(\hat{g}_{ik}\hat{g}_{jl} - \hat{g}_{il}\hat{g}_{jk}) \tag{12}$$

where:

$$\tilde{R}_{ik} = \hat{g}^{jl}\tilde{R}_{ijkl}, \quad \tilde{R} = \hat{g}^{ik}\tilde{R}_{ik} \tag{13}$$

A simple calculation shows then that σ in 9 is equal to:

$$\sigma = \frac{\tilde{S}_{ij}\hat{n}^i\hat{n}^j}{\hat{g}_{kl}\hat{n}^k\hat{n}^l} \tag{14}$$

where:

$$\tilde{S}_{ij} \equiv \tilde{R}_{ij} - \frac{1}{2}\tilde{R}\hat{g}_{ij}, \quad \hat{n}^i = \delta^{ijk}_{123}\hat{f}_{jk} \tag{15}$$

δ^{ijk}_{abc} being the Kronecker tensor of rank six. Therefore the eigen-forms of the Zel'manov-Cattaneo tensors are the duals of the eigen-vectors of the Einstein-like quo-tensor $\tilde{S}_{ij}$ and both objects have the same eigen values σ_a.

Let $\hat{n}^i_a$ be three orthonormal eigen-vectors of $\tilde{S}_{ij}$, with corresponding eigen-values σ_a:

$$\hat{g}_{ij} = \epsilon^a \hat{n}_{ai}\hat{n}_{aj}, \quad \tilde{S}_{ij}\hat{n}^j_a = \sigma_a \hat{n}_{ai}, \quad \hat{n}_{ai} = \hat{g}_{ij}\hat{n}^j_a, \quad \epsilon^a = 1. \tag{16}$$

We shall say that a 3-dimensional quo-tensor $\bar{g}_{ij}$ is a *Principal transform*[2] of the Fermat quo-tensor $\hat{g}_{ij}$ if there exist three functions $c_a(x^\alpha)$ such that the following quo-tensor:

$$\bar{g}_{ij} = \epsilon^a \bar{n}_{ai}\bar{n}_{aj}, \quad \bar{n}_{ai} = c_a \hat{n}_{ai} \tag{17}$$

has the following properties:

- It is independent of time:

$$\partial_t \bar{g}_{ij} = 0 \tag{18}$$

- The Riemannian metric that it defines has constant curvature:

$$\bar{R}_{ijkl} = \kappa(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}) \tag{19}$$

- If $\sigma_a = \sigma_b$ then $c_a = c_b$

- Satisfies the quo-harmonic condition:

$$(\hat{\Gamma}^i_{jk} - \bar{\Gamma}^i_{jk})\hat{g}^{jk} = 0 \tag{20}$$

  where the $\bar{\Gamma}^i_{jk}$'s are the Christoffel connection symbols corresponding to the metric $\bar{g}_{ij}$.

If κ above is zero then the metric $\bar{g}_{ij}$ is euclidean and the last condition 20 implies that cartesian coordinates of this metric, which are the most convenient adapted coordinates to use, are a system of quo-harmonic coordinates

[2]This concept was defined in a more restricted environment in [16].

of the Fermat quo-tensor. If $\kappa \neq 0$ then, even when the congruence is quo-harmonic, quo-harmonic coordinates are not necessarily the most convenient adapted coordinates to use. This is the case for instance when the meta-rigid motion is the standard conformal Killing congruence of reference of a Robertson-Walker model.

To interpret the three frame of reference dependent scalars c_a and directions $\bar{n}_{ai}$ we proceed as follows. Let us consider a light signal, or any other signal, which propagates along a null curve. Let us assume that it leaves a location on a world-line W_1 with adapted coordinates x^i at time t, and it reaches a location on a world-line W_2 with adapted coordinates $x^i + dx^i$ at time $t + dt$ where it bounces back intersecting again the world-line W_1 at an event with coordinates x^i and $t + 2d\tau$. We have then:

$$g_{\alpha\beta}(x^\rho)dx^\alpha dx^\beta \quad \text{or} \quad \epsilon_a(\theta^a)^2 = (\theta^0)^2 \tag{21}$$

with:

$$\theta^0 = u_\alpha dx^\alpha = d\tau, \quad \theta^a = \hat{n}^a_i dx^i, \quad \hat{n}^a_i = \hat{n}_{ai} \tag{22}$$

where $d\tau$ is also:

$$d\tau = \sqrt{\hat{g}_{ij}dx^i dx^j} \tag{23}$$

The distance between W_1 and W_2 according to our interpretation of the constant curvature metric $\bar{g}_{ij}$ is:

$$D = \sqrt{\bar{g}_{ij}dx^i dx^j} \tag{24}$$

Therefore the *round-trip speed* of the signal is:

$$v = \sqrt{\frac{\bar{g}_{ij}dx^i dx^j}{\hat{g}_{ij}dx^i dx^j}} \tag{25}$$

If:

$$dx^i = \bar{k}\beta^a \bar{n}^i_a, \quad \text{with} \quad \bar{n}^i_a = \bar{g}^{ij}\bar{n}_{aj}, \quad \delta_{ab}\beta^a\beta^b = 1 \tag{26}$$

where $\bar{k}$ is any small factor, then we obtain:

$$v^{-1} = \sqrt{\frac{\beta_1^2}{c_1^2} + \frac{\beta_2^2}{c_2^2} + \frac{\beta_3^2}{c_3^2}}, \quad \beta_a = \beta^a \tag{27}$$

If instead we represent the direction of propagation by:

$$dx^i = \hat{k}\gamma^a \hat{n}_a^i, \text{with} \quad \hat{n}_a^i = \hat{g}^{ij}\hat{n}_{aj}, \quad \delta_{ab}\gamma^a\gamma^b = 1 \tag{28}$$

then we obtain the equivalent result:

$$v = \sqrt{\gamma_1^2 c_1^2 + \gamma_2^2 c_2^2 + \gamma_3^2 c_3^2}, \quad \gamma_a = \gamma^a \tag{29}$$

It follows from 27 or 29 that the principal directions $\bar{n}_a$ are the directions where the round-trip speed of a light signal is stationary and that the stationary values are given by the scalars c_a.

2.3 *Chorodesic synchronizations*

. The choice of a foliation to define the synchronization of a frame of reference is to some extent less essential than the choice of a quo-harmonic congruence to define its meta-rigid motion. Besides, time is a much better understood concept in General relativity than the concept of space. The role of a synchronization is to be a first step towards a convenient definition of a universal scale of time, and the properties that are to be required from a synchronization will depend on the use for which it is intended. Atomic and sidereal time are two scales of common use which correspond to different synchronizations of the frame of reference co-moving with the Earth. Not to mention many other scales of time used in astronomy. To illustrate this point we consider briefly how the *International Atomic Time (TAI)* scale is defined. It is based on the definition of the second in the international system of units *SI* as a duration derived from the frequency of a particular atomic transition.

This definition is universal, in the sense that any physicist is able to use a well defined second, but at the same time is local because the identification of atomic time with proper-time duration implies that this definition of the second can be only used to define a scale of time along the world-line of the clock being used used as a standard reference.

To define a scale of time on Earth to be used on navigation systems like the GPS for instance requires a different approach. In this case[3] the second is defined as above with some precisions added. The resulting definition of *TAI* is:

TAI is a coordinate time scale defined in a geocentric reference frame with an SI second realized on the rotating geoid.[4]

It follows from this that the operational definition of the second is relative to some particular locations, i.e. world-lines of a particular congruence, which in this case is the rotating Killing congruence corresponding to the frame of reference co-moving with the Earth. A second at some other location is then defined as to nullify the relativistic red-shift between any pair of clocks of reference co-moving with the Earth. The second at any other location, not in the geoid, is then the interval of time separating the arrival of two light signals sent, one second apart, from a standard clock located on the geoid. The red-shift formulas of General relativity allow then to compare the rhythm of any two clocks that can be joined with light signals.

Let $\mathcal{R}$ be any time-like congruence. A *Chorodesic* C of $\mathcal{R}$ is by definition, [14] [8], a line such that its tangent vector p^α satisfy the following equations:

$$\frac{dx^\alpha}{d\lambda} = p^\alpha, \quad \frac{\nabla p^\alpha}{d\lambda} = \frac{1}{2}u^\alpha \Sigma_{\mu\nu}p^\mu p^\nu, \tag{30}$$

where λ is the proper length along C, and $\Sigma_{\mu\nu}$ is the covariant expression of Born's deformation rate 117. Obviously if $\Sigma_{\mu\nu} = 0$, i.e. if $\mathcal{R}$ is a Born congruence then the chorodesics of $\mathcal{R}$ and the geodesics of the space-time coincide.

Chorodesics of a congruence are important mainly because of the following result:

If a space-like chorodesic C is orthogonal to a world-line of a congruence $\mathcal{R}$ then it is orthogonal to all the world-lines of the congruence $\mathcal{R}$ that it crosses.

This follows from deriving the scalar product $p^\rho u_\rho$ along C. We thus get:

$$\frac{d}{d\lambda}(p^\rho u_\rho) = -\frac{1}{2}\Sigma_{\mu\nu}p^\mu p^\nu + \frac{1}{2}p^\rho p^\sigma(\nabla_\rho u_\sigma + \nabla_\sigma u_\rho) \tag{31}$$

[3]See for instance [17]

[4]The geoid is a level surface of constant geo-potential (gravity and centrifugal potential) at "mean sea level", extended below the continents.

and using the definition of $\Sigma_{\mu\nu}$ this is equivalent to:

$$\frac{d}{d\lambda}(p^\rho u_\rho) = -2(p^\rho u_\rho)(p^\rho \Lambda_\rho) \tag{32}$$

from where the result above follows.

Particular foliations associated with a congruence $\mathcal{R}$ are the one parameter family of hyper-surfaces generated by the chorodesics orthogonal to any particular world-line of the congruence. We call these foliations *Chorodesic synchronizations.* In general a chorodesic synchronization depends on a world-line seed. But if the congruence is integrable then all its chorodesic synchronizations coincide with the family of hyper-surfaces orthogonal to the congruence.

Let $\mathcal{C}$ be the chorodesic synchronization orthogonal to some world-line W of a frame of reference $\mathcal{R}$. Then the associated *Atomic (or Proper) Time Coordination (TAC)* scale is by definition a time coordinate which value at any event E is the proper time interval between some arbitrary event on W and the intersection with W of the chorodesic hypersurface of $\mathcal{C}$ containing E.

Notice that, even when the congruence of reference $\mathcal{R}$ is hypersurface orthogonal, i.e. there exists a single chorodesic synchronization, the TAC may depend on the particular world-line W which is used to identify the coordinate time with proper-time along W. Switching from a world-line seed of a TAC to another gives rise to a sub-group of adapted coordinate transformations that we shall call the *TAC sub-group.* An explicit example will be shown in section 6. A complete definition of the TAC requires therefore to choose a particular world-line $\mathcal{S}$, or eventually a bunch of equivalent ones. On Earth the TAC, i.e. the TAI, is associated to the bunch of world-lines $\mathcal{S}$ of locations on the geoid.

We explain below in geometrical terms why this allows to define a convenient TAC, in any stationary frame of reference, not necessarily co-moving with the Earth. Let us assume that $\mathcal{R}$ is a Killing congruence. In this particular case the TAC with seed on any particular world-line of $\mathcal{R}$ implements on any other world-line of $\mathcal{R}$ the local time scale correctly red-shifted. In fact in this case the chorodesics of $\mathcal{R}$ are geodesics, and the flow of geodesics is invariant under the group of motions generated by the Killing vector field of the Killing congruence. Let A_1 and B_1 be two events on a world-line W_1 of $\mathcal{R}$ and let A_2 and B_2 be the events at which the geodesic hyper-surfaces

orthogonal to W_1 intersect a second world-line W_2 of the congruence. Let L_2 and M_2 be the intersections of the future (resp. past) light-cones with origins A_1 and B_1 with W_2. It follows then very easily that the proper-time interval $A_2 - B_2$ on W_2 is the same as $L_2 - M_2$.

If $\mathcal{R}$ is not a Killing congruence then the preceding result does not hold, nevertheless general TAC scales, as we defined them, remain useful as theoretical time references. Its practical use remains to be understood.

3 Elastic deformations

The main ingredients in the theory of elastic deformations are:

- 1.- A definition of the infinitesimal displacement vector.
- 2.- The kinematical description of the strain tensor field as a measure of the deformation relative to some idealized rigid body of reference.
- 3.- The dynamics of the stress tensor and the equations of equilibrium
- 4.- The Hooke's law
- 5.- The Beltrami-Michell integrability conditions

In ordinary elasticity theory items 1 and 2 rely heavily on the geometry of euclidean space. First of all in using the concept of rigid motion as a motion of reference and second in measuring strains in terms of change of euclidean distances between corresponding points of the initial body and the deformed one. Some of the definitions below make use of the concepts introduced in the preceding section as substitutes to the elementary ones used in ordinary elasticity theory.

Let us consider an elastic body which motion is described by a congruence $\mathcal{B}$ with unit tangent vector-field w^α. Suppose that among the motions of reference, i.e. the meta-rigid motions of the space-time we are considering, there is one $\mathcal{R}$, such that i) One of the world-lines W of $\mathcal{R}$ coincides with one of the world-lines of $\mathcal{B}$ and the rotation rates of both congruences along W coincide, ii) there exists a one to one correspondence between the world-lines of $\mathcal{B}$ and a subset of world-lines of $\mathcal{R}$ such that the orthogonal distance, in the sense of the principal transform metric $\bar{g}_{ij}$ of the Fermat tensor, between

corresponding world-lines, remains small. Then we shall say that the motion of $\mathcal{B}$ is a deformation of $\mathcal{R}$ and define this kinematic deformation in terms of the geometry of the meta-rigid motion $\mathcal{R}$.

Let us consider a system of coordinates adapted to $\mathcal{R}$ and let $\mathcal{B}$ be defined in this system of coordinates by the parametric equations:

$$y^\alpha = x^\alpha + \zeta^\alpha(x^\rho) \tag{33}$$

x^0 being the parameter. A short and explicit calculation proves that if u^α is the unit tangent vector to the congruence $\mathcal{R}$ and we use the following notation:

$$w^\alpha = u^\alpha + \delta u^\alpha \tag{34}$$

then one has in this system of adapted coordinates:

$$\delta u^0 = -\xi^{-1}\varphi_i\partial_0\zeta^i, \quad \delta u^i = \xi^{-1}\partial_0\zeta^i, \tag{35}$$

a result which can be written in the manifestly covariant form:

$$\delta u^\alpha = \hat{g}^\alpha_\lambda \mathcal{L}(\zeta)u^\lambda \tag{36}$$

where $\mathcal{L}()$ is the Lie derivative operator. It follows from this result that the generator of a deformation is defined up to the transformation:

$$\zeta^\alpha \rightarrow \zeta^\alpha + ku^\alpha \tag{37}$$

where k is any function. This property can thus be used to assume, without any loss of generality, that u^α and ζ^α are orthogonal: $u_\alpha\zeta^\alpha = 0$. Or in adapted coordinates:

$$\zeta_0 = 0 \tag{38}$$

By definition the strain tensor of the congruence $\mathcal{B}$ with respect to the frame of reference $\mathcal{R}$ is, using a system of coordinates adapted to it, the tensor:

$$\epsilon_{ij} = \bar{\nabla}_i\zeta_j + \bar{\nabla}_j\zeta_i \tag{39}$$

where:

$$\bar{\nabla}_i \zeta_j = \hat{\partial}_i \zeta_j - \bar{\Gamma}^k_{ij} \zeta_k \tag{40}$$

This definition is strongly dependent on the theory of the frames of reference presented in the preceding section. Notice also that ϵ_{ij} defines ζ_i only up to a transformation:

$$\zeta_\alpha \to \zeta_\alpha + s_\alpha \tag{41}$$

where s_α is a solution of the following equations:

$$\bar{\nabla}_i s_j + \bar{\nabla}_j s_i = 0 \tag{42}$$

If the deformation is a *static* one, this meaning that $\delta u^\alpha = 0$ in 34, then from 35 it follows that:

$$\partial_0 \zeta^i = 0 \tag{43}$$

If moreover $\bar{g}_{ij} = \hat{g}_{ij}$, i.e. if the Fermat tensor is a Riemannian metric with constant curvature, then the definition of strain in 39 is the usual one of ordinary elasticity theory, and Eqs. 42 are the equations defining the infinitesimal generators of ordinary rigid motions of classical physics. This behavior in these particular cases is our justification for the definition 39.

The stress-energy of an elastic test body is:

$$T^{\alpha\beta} = \rho w^\alpha w^\beta - \pi^{\alpha\beta}, \quad \pi^\alpha_\beta w^\beta = 0 \tag{44}$$

where ρ is its density, w^α is its unit four-velocity field, and where $\pi_{\alpha\beta}$ is the tensor describing the stresses which are present as a result of the volume forces acting on the body and those acting on its surface. Its components have to be considered to have the same order of smallness as that of δu^α and therefore the second Eq. 44 is equivalent to $\pi_{\alpha\beta} u^\alpha = 0$. The generalization of the equilibrium equations of ordinary elasticity theory to a relativistic one is then, assuming that no other forces that gravitational or inertial ones act on the body:

$$\nabla_\alpha T^\alpha_\beta = 0, \tag{45}$$

This ingredient is common to any relativistic elasticity theory and owes nothing to any theory about the frames of reference.

The formulation of Hooke's law, which is the next ingredient needed in the theory, depends again from a conceptual point of view on the the choice of the metric used to compare distances between points of a body and the same body after, or during, the action of some additional stresses. Our present version of Hooke's law for homogeneous and isotropic bodies is:

$$\epsilon_{ij} = \frac{1}{3(3\lambda+2\mu)}\pi\bar{g}_{ij} + \frac{1}{2\mu}(\pi_{ij} - \frac{1}{3}\bar{g}_{ij}\pi), \quad \pi = \bar{g}^{ij}\pi_{ij} \tag{46}$$

where λ and μ are the Lamé's parameters of the body.

The Beltrami-Michell equations, which is the last ingredient we mentioned at the beginning of this section, are the equations that should be imposed to the stress-tensor π_{ij} to satisfy the integrability conditions corresponding to the definition 39 of the strain tensor, the conservation equations 45, and Hooke's law. They are cumbersome and they will not be written here.

In [18] the Fermat tensor instead of a principal transformation of it was used in Eqs. 39 and 46. The choice which we have made here agrees better with our present confidence on the meaning and the necessity of considering the metric $\bar{g}_{ij}$ to implement the free mobility property of idealized rigid bodies.

4 Local anisotropy of space in a frame of reference co-moving with the Earth

We consider in this section[5] the linearized line-element of the gravitational field of the earth in a co-moving frame of reference, and we construct the euclidean principal transformation of the corresponding Fermat tensor. This application is particularly important because it leads to a prediction that can be tested with available technology.

The line-element of the linearized exterior gravitational field of a non-rotating spheroidal body with mass M, mean radius R, and reduced quadrupole moment J_2 is:

[5] This section has been included here for completeness. It is an abstract of a longer contribution by Ll. Bel and A. Molina to this same volume. A full version of this work can be seen also in [19]. We use natural units such $G = c = 1$

$$ds^2 = -(1 - 2U_G)dt^2 + (1 + 2U_G)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)) \tag{47}$$

where U_G is:

$$U_G = \frac{M}{r}\left(1 + \frac{1}{2}\frac{J_2 R^2(1 - 3\cos^2\theta)}{r^2}\right) \tag{48}$$

If the body is rotating with angular velocity Ω then we obtain the line-element substituting in 47 $\varphi + \Omega t$ for φ. The line element of the Earth at the approximation we are interested in is therefore:

$$ds^2 = -(1 - 2(U_G + U_\Omega))dt^2 + 2A_\varphi dt d\varphi + (1 + 2U_G)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)) \tag{49}$$

where:

$$U_\Omega = \frac{1}{2}\Omega^2 r^2 \sin^2\theta, \quad A_\varphi = \Omega r^2 \sin^2\theta \tag{50}$$

The Fermat line-element of space can then be written as:

$$d\hat{s}^2 = (\hat{\theta}^1)^2 + (\hat{\theta}^2)^2 + (\hat{\theta}^3)^2 \tag{51}$$

where:

$$\hat{\theta}^1 = (1+U_G)(dr + J_2 f r d\theta), \ \hat{\theta}^2 = (1+U_G)(-J_2 f dr + r d\theta), \ \hat{\theta}^3 = (1+U) r \sin\theta d\phi \tag{52}$$

with:

$$f = -4\frac{R^2}{r^2}\left(1 - q\frac{r^5}{R^5}\right)\sin\theta\cos\theta \quad \text{and} \quad q = \frac{1}{4}\frac{\Omega^2 R^3}{M J_2} \tag{53}$$

are the principal forms of the its Ricci tensor.

To construct the Principal transform of the Fermat quo-tensor as we defined it in subsection 2.2 we need to find the three scalars c_a such that the metric:

$$d\bar{s}^2 = c_1^2(\hat{\theta}^1)^2 + c_2^2(\hat{\theta}^2)^2 + c_3^2(\hat{\theta}^3)^2 \tag{54}$$

satisfies the two sets of equations 19, with $\kappa = 0$, and 20. These coefficients have been found to be at the desired approximation:

$$c_1 = 1 - \frac{M}{r}\left(1 + \frac{1}{5}\frac{R_1^2}{r^2}\right) + \frac{MJ_2R^2}{r^3}\left(6 + \frac{6}{5}\frac{R^2}{r^2} - \left(5 + \frac{9}{5}\frac{R^2}{r^2}\right)\sin^2\theta\right)$$
$$+\Omega^2 r^2\left(-\frac{27}{5} + \frac{3}{10}\frac{R^2}{r^2} + \left(\frac{15}{2} - \frac{3}{10}\frac{R^2}{r^2}\right)\sin^2\theta\right) \quad (55)$$

$$c_2 = 1 - \frac{M}{2r}\left(3 - \frac{1}{5}\frac{R_2^2}{r^2}\right) + \frac{MJ_2R^2}{r^3}\left(1 - \frac{3}{5}\frac{R^2}{r^2} - \left(\frac{19}{4} - \frac{21}{20}\frac{R^2}{r^2}\right)\sin^2\theta\right)$$
$$+\Omega^2 r^2\left(-\frac{19}{5} + \left(\frac{13}{2} + \frac{3}{10}\frac{R^2}{r^2}\right)\sin^2\theta\right) \quad (56)$$

$$c_3 = 1 - \frac{M}{2r}\left(3 - \frac{1}{5}\frac{R_2^2}{r^2}\right) + \frac{MJ_2R^2}{r^3}\left(1 - \frac{3}{5}\frac{R^2}{r^2} - \left(\frac{9}{4} + \frac{3}{4}\frac{R^2}{r^2}\right)\sin^2\theta\right)$$
$$+\Omega^2 r^2\left(-\frac{19}{5} + 4\sin^2\theta\right) \quad (57)$$

where R_1 and R_2 are in this approach two free parameters of the order of the mean radius R of the Earth. The values of these coefficients provide the basis for a fresh new discussion about the anisotropy of space in the neighborhood of the surface of the Earth and the means to measuring it.

From our general principle of interpretation of subsection 2.2 according to which the coefficients c_a are in particular the velocities of light along the principal directions 52 and from 29, it follows at the required approximation, that the velocity of light in the direction with components γ_a with respect to the principal triad 52 is:

$$v = 1 + \gamma_1^2\alpha_1 + \gamma_2^2\alpha_2 + \gamma_3^2\alpha_3 \quad \text{with} \quad \alpha_a = 1 - c_a \quad (58)$$

Since the interferometer in the Michelson-Morley experiment is kept horizontal the relevant expression in this case is:

$$v = 1 + \alpha_2\cos^2 A + \alpha_3\sin^2 A = 1 + \frac{1}{2}(\alpha_2 + \alpha_3) + \frac{1}{2}(\alpha_2 - \alpha_3)\cos 2A \quad (59)$$

where A is the azimuth of the direction γ_a, and where in particular the quantity which measures the anisotropy is:

$$a_2 = \frac{1}{2}(c_2 - c_3) = -\frac{1}{10}\left(\frac{11MJ_2}{R} - 14\Omega^2R^2\right)\sin^2\theta \tag{60}$$

on the surface of the Earth at the colatitude θ. Taking into account the following values: $M = 0.00444$ m, $R = 6378164$ m, $\Omega = 2.434 \times 10^{-13}$ m^{-1}, and $J_2 = 0.0010826$ we have:

$$a_2 = 2.5 \times 10^{-12}\sin^2\theta \tag{61}$$

This dependence on the colatitude is a signature that it will help in testing this effect.

Up to know only one experiment, [20], has been performed with a sufficient sensitivity to test this result. In this experiment the coefficients a_2 and b_2 in the following expression:

$$v = \frac{1}{2}a_0 + a_2\cos 2A + b_2\sin 2A \tag{62}$$

are measured, but since the paper does not indicate what was the experimental origin of the angle A only the value of $\sqrt{(a_2^2 + b_2^2)}$ can be considered as meaningful. The result which is mentioned is 2.1×10^{-13} at a colatitude of $50°$. The corresponding result derived from 61 is 1.5×10^{-12}. The agreement here is only qualitative. A better one is obtained with a slightly different approach (See the footnote at the beginning of this section and the references therein). In our opinion it is an urgent task that somebody repeats this experiment with the required sensitivity.

5 The definition of the quantum vacuum in Milne's universe

The two preceding applications made an explicit use of the interpretation we proposed in section 2 embodied in the concept of meta-rigid motions. They made use also of principal transformations of the Fermat tensor into a constant curvature metric, thus implementing the property of free mobility of idealized rigid bodies. Neither application necessitated the use of any

particular synchronization, contrary to the application of this section and the two following ones.

We consider here the problem of defining the quantum vacuum of a quantized scalar field in Cosmology. But instead of keeping the subject rather general, as it was done in [21], we describe here the problem in the context of Milne's universe, because its geometry, which is locally flat, raises some other problems related to the main subject of this contribution,[6].

Let us consider Minkowski's space-time $\mathcal{M}$ referred to a galilean frame of reference and cartesian coordinates x^α. Let E be an event on it and let us consider the interior of the future-pointing light-cone with vertex E. This defines the manifold $\mathcal{U}$ of Milne's universe. Its metric being that of the flat space-time corresponding to its immersion on $\mathcal{M}$. Let us consider on $\mathcal{U}$ the congruence $\mathcal{R}$ of time-like geodesics passing through E. An appropriate parameterization of $\mathcal{R}$ is:

$$x^0 = t, \quad x^i = z^i H t \tag{63}$$

where H is an arbitrary constant with dimensions T^{-1}. Using (t, z^i) as adapted coordinates the line element of flat space-times becomes:

$$ds^2 = -(1 - H^2 r^2) dt^2 + H^2 \delta_{ij} (2tz^i dt + t^2 dz^i) dz^j \tag{64}$$

where $r^2 = \sum (z^i)^2$. The following time-gauge transformation:

$$t \to t(1 - H^2 r^2)^{1/2} \tag{65}$$

and the use of polar coordinates instead of cartesian ones brings the line element 64 to the following form:

$$ds^2 = -dt^2 + d\hat{s}^2, \quad d\hat{s}^2 = H^2 t^2 d\bar{s}^2 \tag{66}$$

with:

$$d\bar{s}^2 = \frac{1}{1 - H^2 r^2} \left(\frac{dr^2}{1 - H^2 r^2} + r^2 (d\theta^2 + \sin^2 d\varphi^2) \right) \tag{67}$$

which after the final transformation of the coordinate r:

[6] Another particular example can be seen in [22]

$$r \rightarrow r(1-H^2r^2)^{-1/2} \tag{68}$$

becomes:

$$d\bar{s}^2 = \frac{dr^2}{1-H^2r^2} + r^2(d\theta^2 + \sin^2 d\varphi^2) \tag{69}$$

We recognize in 66 with $d\bar{s}^2$ given by 69 one of the familiar forms of a Robertson-Walker metric. In this case, i.e. the Milne's universe, the scale factor is Ht and the space curvature is $-H^2$.

Since the Laplacian $\hat{\triangle}$ defined in 7 coincides in this case with $\bar{\triangle}$, the Laplacian corresponding to the time independent principal transform metric 69, it follows that the congruence $\mathcal{R}$ is quo-harmonic. Although quo-harmonic coordinates are not in this case the more convenient coordinates to use, even in its polar form.

Notice also that since the family of hyper-surfaces $t = const$ is orthogonal to $\mathcal{R}$ the synchronization that it defines is a chorodesic one. It is not a geodesic one because Born's deformation rate of $\mathcal{R}$ is not zero. The time scale is the corresponding TAC as we defined it in section 2. In this case it does not depend on the world-line of $\mathcal{R}$ to which it is associated.

It is not important here to decide whether or not Milne's universe is a good model of our real universe. It is sufficient to recognize that a real universe could have existed for which Milne's universe would have been an acceptable model. Two remarks are relevant here: i) the first one is that although the local geometry of both the Milne's and Minkowski's universe is the same their physical meaning is quite different. And anybody, who would say that they describe the same physical reality on the basis that the coordinate singularity at $r = 0$ could be avoided by extending the geodesically incomplete Milne's universe to span the full Minkowski's universe, would mathematically be correct but absolutely wrong from a physical point of view. ii) the second remark is that even from the local point of view Milne's and Minkowski's universes describe different physical realities if the theory of frames of reference that we presented in section 2 is accepted. In fact one has to consider that the congruence $\mathcal{R}$ which we considered to build Milne's model is a perfect meta-rigid motion in this universe while it is not in Minkowski's one.

The Klein-Gordon equation in Minkowski's space-time referred to a galilean frame of reference is :

$$(-\partial_t^2 + \triangle - \frac{m^2}{\hbar^2})\psi = 0 \tag{70}$$

where $\hbar$ is the reduced Plank's constant, m is the mass of the quantum of the field and $\triangle$ is the Laplacian corresponding to the euclidean metric.

The positive(negative) energy modes are the solutions of 73 which have the following form:

$$\varphi_\epsilon(x^\alpha, \vec{k}) = (2\pi\hbar)^{-3/2} u_\epsilon(t, \vec{k}) \exp(\frac{i}{\hbar}\vec{k}\vec{x}), \tag{71}$$

with:

$$u_\epsilon(t, \vec{k}) = (2\omega)^{-1/2} \exp(-\frac{i}{\hbar}\epsilon\omega t), \quad \omega(\vec{k}) = +(\vec{k}^2 + m^2)^{1/2} \tag{72}$$

where $\epsilon = 1\,(-1)$. This decomposition of the space of solutions in two distinct sub-spaces is complete, intrinsic and it is what is needed to introduce the concept of particles and anti-particles as quanta of a quantized scalar field. It is usually referred to as its quantum vacuum.

In Milne's universe the Klein-Gordon equation for a classical field Ψ is:

$$(-\partial_t^2 - 3\dot{\sigma}\partial_t + e^{-2\sigma}\bar{\triangle} - \frac{m^2}{\hbar^2})\psi = 0 \tag{73}$$

where $\sigma = \ln(Ht)$ and $\bar{\triangle}$ is the laplacian of the constant curvature metric $d\bar{s}^2$. A complete set of solutions of 73 can be obtained in two steps: i) separating first the time dependence and the space dependence, i.e. by looking for solutions of the following form:

$$\Psi(x^\alpha) = u(t)\psi(x^i, \vec{k}) \tag{74}$$

$\psi(x^i, k^2)$ being a complete set of eigen functions of the Laplace operator $\bar{\triangle}$ of the metric 69:

$$\bar{\triangle}\psi(x^i, \vec{k}) = -(k^2/\hbar^2)\psi(x^i, \vec{k}) \tag{75}$$

and ii) splitting the space of solutions $u(t, k^2)$ of the following equation:

$$\hbar^2\ddot{u} + 3\hbar^2 t^{-1}\dot{u} + \omega^2 u = 0, \quad \omega^2 = \frac{k^2}{H^2 t^2} + m^2 \tag{76}$$

in two groups. To do that we proposed in [21] to solve the problem in two steps: a) to solve first the Riccati equation:

$$i\hbar\dot{f} + f^2 + 3i\hbar t^{-1} f - \omega^2 = 0 \tag{77}$$

and b) to solve after that the equation:

$$i\hbar\dot{u} = fu \tag{78}$$

f being a solution of Eq. 77. This guarantees that the first order equation 78 is an order reduction of 76 in the sense that every solution of 78 is a solution of this equation but not vice versa. The positive energy solutions of 76 are then by definition the solutions of 78 with f being the solution of 77 satisfying the following limit condition with $\epsilon = +1$

$$\lim_{\hbar \to 0} f_\epsilon = \epsilon\omega. \tag{79}$$

The negative energy solutions are those corresponding to $\epsilon = -1$

If $m = 0$ the solutions are very easy to obtain. They are

$$f_\epsilon = bt^{-1} \quad \text{with} \quad b = -i\epsilon\hbar + (-\hbar^2 + k^2/H^2)^{1/2} \tag{80}$$

And we have thus:

$$u_\epsilon(t, \vec{k}) = A(t_0, \vec{k}) \exp(-\frac{ib}{\hbar}\ln(t/t_0)) \tag{81}$$

where t_0 is an arbitrary constant and A is a function of t_0 and k^2 to be fixed with a normalization condition.

As we see the quantum vacuum of Minkowski's universe referred to a galilean frame of reference is quite different from that of Milne's universe referred to the frame of reference $\mathcal{R}$ defined above. Nevertheless it makes sense to compare 81 to 72 (with $m = 0$) because in either case the frame of reference which is used correspond to the same definition: The meta-rigid motion is a quo-harmonic congruence, the synchronization is a chorodesic one, and the time scales are similarly defined.

6 The missing group

In 1872 Felix Klein gave a celebrated definition saying that a geometry is a theory of the invariants under a particular group of transformations. The geometry of both Classical physics and Special relativity will fit this definition because the laws of physics in both cases are invariant under the group of Galileo or under the Poincaré group respectively. But, which is the group of transformations defining the geometry of space-time as we use it in general relativity ?. A possible honest answer is to to think about the isometry group of the space-time metric and say that in general there is none, or that when there is one it is usually very small and does not fit very well in Klein's definition. Does this mean that there is a missing group? There is of course also the possibility of saying that there is no missing group and that one has to think of the group of diffeomorphisms of the space-time manifold as a group of symmetry, which in our opinion it is not. I would like to insist here on a suggestion I made a few years ago in [14] by saying that for any space-time we should find a group more general than the group of isometries and more meaningful than the group of diffeomorphisms.

At the end of this section we describe a non trivial example illustrating this idea. The first part of the idea consists in realizing that the more fundamental group of classical mechanics is not the Galileo group but a much larger non-Lie group, namely the group of rigid motions which transform the cartesian and time coordinates of two rigid frames of reference one into another:

$$t' = t - A^0, \qquad x'^i = R^i_j(t)\left[x^j - A^j(t)\right], \tag{82}$$

where $R^i_j(t)$ is a rotation matrix. This group contains as sub-groups the euclidean group in three dimensions:

$$x'^i = R^i_j(x^j - S^j), \qquad R^i_j, S^j : Constants \tag{83}$$

and the inhomogeneous Galileo group:

$$t' = t - A^0, \quad x'^i = R^i_j\left(x^j - V^j t - A^j\right), \qquad R^i_j, V^j, A^\alpha : Constants \tag{84}$$

which are the sub-groups people usually think of when considering the basic symmetries in classical physics.

Let us consider the class $\mathcal{N}$ of space-times for which there exists a system of coordinates such that their line-element can be written as:

$$ds^2 = -(1-2U)dt^2 + 2U_i dt dx^i + \delta_{ij} dx^i dx^j \tag{85}$$

where U and U_i are functions of t and x^i. The group of rigid motions acts on a line-element of this sort as a group of generalized isometries in the sense that it maps the class $\mathcal{N}$ into itself, keeping invariant the restriction of the line-element on any hyper-surface $t = const.$. Our quest for the missing group stars with this remark which led also in [4] to a generalized Newtonian theory of gravitation in classical mechanics for which the meaningful covariance is the generalized invariance under the group of rigid motions.

Let us consider now a given space-time M and let us assume that we know the class $\mathcal{C}$ of quo-harmonic congruences satisfying every necessary condition to qualify as meta-rigid motions of M. This class may contain some, or all, of the isometries of M; some, or all, of Born's rigid congruences. We wrote: some or all, because global conditions or physical requirements may disqualify some quo-harmonic congruences as meta-rigid motions. In general $\mathcal{C}$ will contain also an infinite number of other congruences, possibly labeled by arbitrary functions and parameters. Let us assume also that for each of these congruences the principal transformation of the Fermat tensor is unambiguously defined by every necessary global condition and therefore there exists a well defined system of quo-harmonic coordinates z^i corresponding to the constant curvature image of the Fermat tensor. Let us assume finally that a chorodesic synchronization has been chosen for each congruence of $\mathcal{C}$ and therefore the time coordinate $z^0 = \tau$ is sufficiently well defined, up to a time sub-group transformation. If $\mathcal{R}$ and $\mathcal{R}'$ are two meta-rigid motions of $\mathcal{C}$ labeled by L and L' we may consider the coordinates (τ', z'^i) of an event with respect to $\mathcal{R}'$ as functions of the coordinates (τ, z^i) of the same event with respect to $\mathcal{R}$. We write this coordinate transformation symbolically:

$$z'^\alpha = \varphi^\alpha_L(z^\beta, L') \tag{86}$$

By definition this family of transformations satisfy a composition law:

$$\varphi^\alpha_{L'}(\varphi^\beta_L(z^\gamma, L'), L'') = \varphi^\alpha_L(z^\beta, \lambda(L, L', L'')) \tag{87}$$

where $\lambda(L, L', L'')$ is some composition law in the space of labels.

We analyze below with more detail the family of transformations 86 in a particular case. Let M be Minkowski's space-time with line element in a system of galilean coordinates x^α:

$$ds^2 = -dt^2 + \delta_{ij} dx^i dx^j. \tag{88}$$

Let W be a time-like world-line with parametric equations, and unit time-like tangent vector:

$$x^\alpha = y^\alpha(\tau), \quad w^\alpha(\tau) = \dot{y}^\alpha(\tau) \tag{89}$$

where τ is proper-time along W measured from a fixed origin on W with galilean coordinates y_0^α.

There is one and only one irrotational Born congruence containing W and this is the Fermi-congruence with base-line W and parametric equations:

$$x^\alpha = e_i^\alpha(\tau) z^i + y^\alpha(\tau), \tag{90}$$

where $e_i^\alpha(\tau)$ is a triad of orthonormal vector fields, orthogonal to $w^\alpha(\tau)$, Fermi propagated along W, i.e. satisfying the differential equations:

$$\dot{e}_i^\alpha = -w^\alpha b_\rho e_i^\rho \quad b_\rho = \dot{w}_\rho. \tag{91}$$

Each congruence of $\mathcal{B}$ can be labeled by three functions of one argument: $y^i(\tau)$ and one parameter: $y^0(0)$. Since if we know $y^i(\tau)$ we know also $u^i(\tau) = \dot{u}^i$, and since u^α has to be unitary this means then that we know also u^0. Integrating the differential equation $\dot{y}^0 = u^0$ with the initial condition $y^0(0)$ yields then the function $y^0(\tau)$ and the parametric equations of the base-line of the congruence are known. This proves that the family of congruences $\mathcal{B}$ has the same number of degrees of freedom: three arbitrary functions $y^i(\tau)$ of one variable and a parameter $y^0(0)$, as the family of irrotational rigid motions of classical physics.

Differentiating Eqs. 90 and substituting the result in 88 the line element of M becomes [24]:

$$ds^2 = -[1 + a_k(\tau) z^k]^2 d\tau^2 + \delta_{ij} dz^i dz^j, \quad a_k(\tau) = e_k^\alpha(\tau) b_\alpha(\tau). \tag{92}$$

$a_k(\tau)$ is the curvature, i.e. the intrinsic acceleration, of W which parametric equations in the adapted coordinates are $z^i = 0$.

The form of the line-element 92 proves that the Fermat tensor is euclidean, meaning that the principal transformation in this case is trivial; it proves moreover that the Fermi coordinates of space are the corresponding quo-harmonic, in this case cartesian, coordinates. On the other hand the $\tau = const.$ hyper-surfaces define a chorodesic, in this case geodesic, synchronization.

In [14] we defined a compound description of a metric $g_{\alpha\beta}(z^\rho)$ as a description of this metric by 10 functions $\bar{g}_{\alpha\beta}(x^\rho, f_{(L)})$ where $f_{(L)}$ is a set of functions of x^ρ belonging to a restricted functional space $\mathcal{F}$. The metric 92 is an example of compound description of the Minkowski metric where the functional space $\mathcal{F}$ is the set of the triads $a_k(\tau)$ of functions of one variable.

We shall say that the class of congruences $\mathcal{C}$ is isometrically complete if a compound description of the space-time metric exist such that we have:

$$g_{\alpha\beta}(x^\rho; L) = \bar{g}_{\alpha\beta}(x^\rho; f_L(x^\sigma)) \tag{93}$$

where for each L, i.e. for each quo-harmonic congruence of $\mathcal{C}$, x^ρ is an admissible system of quo-harmonic coordinates[7] and $f_{(L)}(x^\sigma)$ is an appropriate set of $\mathcal{F}$. By construction the class $\mathcal{B}$ of irrotational Born rigid motions in Minkowski's space-time 90 is isometrically complete and the corresponding compound description of the space-time metric is 92

The introduction of the concept of compound description of a space-time metric allows to interpret the coordinate transformations 86 as a group of transformations defining what we called in [14] *Generalized isometries*. In fact, let us consider the family of coordinate transformations that leave invariant the compound description of an isometrically complete class of quo-harmonic congruences $\mathcal{B}$. This family of transformations, which coincides with the family of transformations 86, is a group, actually an infinite dimensional one in general. It must be understood though that the space on which these transformations act as a group is the product $M \times \mathcal{F}$, or some open neighborhood of it. Underlining this it may be convenient to use the notation:

$$z'^\alpha = \psi^\alpha(z^\beta, f_{(L)}; \Lambda) \quad f_{(L')} = \chi_{(L')}(f_{(L')}, \Lambda) \tag{94}$$

where Λ designates a functional space of parameters.

[7] We call also quo-harmonic a time coordinate of the TAC scale corresponding to the frame of reference being considered

In [14] we defined the *Born's group* as being the group above for the class of Born congruences $\mathcal{B}$ in Minkowski's space-time. In other words the Born's group is the group of transformations that leave invariant the compound description 92.

For each set of functions $a_k(\tau)$ the family of transformations that leave invariant the compound description and the functions $a_k(\tau)$ themselves, defines a realization of the Poincaré group.

A particular sub-group of the Born group deserves a particular attention. It is the *Time sub-group* which leaves invariant a particular congruence of $\mathcal{B}$ without leaving invariant the set of functions $a_k(\tau)$. It is the group of transformations of Fermi coordinates that switch from a world-line seed of the congruence to another. The analytic realization of this finite dimensional sub-group is:

$$z'^i = z^i - \lambda^i, \quad \tau' = \tau + v_k(\tau)\lambda^k, \quad a_k'(u, \lambda^i) = \frac{a_k(u)}{1 + a_j(u)\lambda^j} \tag{95}$$

where:

$$v_k(\tau) = \int_0^\tau a_k(u)du \tag{96}$$

7 The froth of the universe

Let us consider a general Robertson-Walker metric:

$$ds^2 = -c^2dt^2 + F^2(t)d\bar{s}^2, \quad d\bar{s}^2 = N^2(r)\delta_{ij}dx^i dx^j, \tag{97}$$

where:

$$N(r) = \frac{1}{1 + kr^2/4}, \quad r = \sqrt{\delta_{kl}x^k x^l}. \tag{98}$$

We shall assume that the scale factor F is dimensionless, that t has dimensions of time T and that r has dimensions of space L. The curvature constant k will have therefore dimensions L^{-2} and the universal constant c[8] dimensions LT^{-1}.

The line-element 97 can be written also in the following form:

[8]In this section we do not use normalized units

$$d\tau^2 = dt^2 - \frac{1}{c_{eff}^2} d\bar{s}^2 \quad \text{with} \quad c_{eff} = c/F(t) \tag{99}$$

which suggests that any local process, both in time and in space, will evolve as if the Universe were flat with an effective speed of light equal to c_{eff}. It suggests also the idea of giving different names to the universal constant c and to c_{eff}. Several years ago, [4], we proposed to call Römer's constant the universal constant c which allows the conversion of space to time and vice versa. Instead c_{eff} will be here the speed of those processes mediated by zero mass particles, in particular the propagation of light.

Let us consider the congruence $\mathcal{R}$ with world-lines $x^i = const.$. The Fermat metric of this congruence is:

$$d\hat{s}^2 = (1/c_{eff})^2 d\bar{s}^2 \tag{100}$$

By the same arguments we gave in section 5 it follows from 99 that $\mathcal{R}$ is a quo-harmonic congruence, that the family of hyper-surfaces $t = constant$ define a chorodesic synchronization, and that the coordinate t is a TAC coordinate for every world-line of $\mathcal{R}$.

According to the definition we gave in section 2 the principal transformation of 100 is the metric $d\bar{s}^2$ defined above and therefore the three coefficients c_a are all equal to c_{eff}. The local heuristic interpretation of this function given above is therefore substantiated by the theory of principal transformations, becoming a global and theoretically justified interpretation.

If we assume to know the scale Factor $F(t)$ and the remaining parameters of the model then Einstein's field equations:

$$S_{\alpha\beta} - \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^2} T_{\alpha\beta} \tag{101}$$

will allow to calculate $T_{\alpha\beta}$. The energy-momentum tensor thus defined is however a rather polymorphic object that can be written as if we were dealing with a perfect fluid:

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p\hat{g}_{\alpha\beta}, \quad \hat{g}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \tag{102}$$

where the pressure, p, and the mass density, ρ, are functionals of $F(t)$, but it can be written also as if we were dealing with a scalar field source:

$$T_{\alpha\beta} = \partial_\alpha \Phi \partial_\beta \Phi - g_{\alpha\beta}[\partial_\mu \partial^\mu \Phi - V(\Phi)] \tag{103}$$

with Φ and $V(\Phi)$ being some other functionals of F. It is also possible, among many other possibilities, to think of $T_{\alpha\beta}$ as being a superimposition of expressions of the type 102 and 103.

Of course a cosmologist works the other way around. He wants to derive F from a single polymorphic instance of $T_{\alpha\beta}$ or from a few ones, depending on the stage of the evolution of the universe he considers or the degree of 'realism' he wants to include in his model. The so called Standard model assumes that the history of the universe can be divided in two epochs. It assumes, as the Bible, that at the beginning God said: *Fiat lux*, and the universe became filled with radiation, or equivalently that $T_{\alpha\beta}$ took the form 102 with the equation of state $p = 1/3\rho c^2$. It assumes also that during the, still lasting, second epoch the Universe became matter dominated which means that $T_{\alpha\beta}$ still has the form 102 but this time with $p = 0$. The so called Inflationary models are a variant of the Standard model where one assumes that at some epoch the Universe was dominated by some scalar field and therefore $T_{\alpha\beta}$ could be interpreted as in 103. All the details about this scalar field and the mechanisms of transition from one epoch to another rely heavily on our knowledge of many other branches of physics. This makes these models rather 'realistic' but at the same time they are probably too cumbersome and unnecessarily complicated for what observational cosmology has to offer today.

In [25] we presented a model which is conceptually and technically very simple and seems to offer many of the properties of more elaborate models. The model is based on what we said before at the beginning of this section about the meaning of c_{eff}. The idea consists in observing that the equation of state $p = 1/3\rho c^2$ is the equation of state of a gas of zero mass particles, and that this equation is derived in the frame-work of special relativistic kinetic theory, assuming that the box containing the gas is at rest with respect a galilean frame of reference. In which case the speed of light in vacuum coincides with the universal Römer's constant. The distinction we made above between these two concepts, the heuristic argument we mentioned, and a more detailed analysis included in [25] suggest that the equation of state of such a gas should be instead:

$$p = \frac{1}{3}\rho c^2/F^2 = 1/3\rho c_{eff}^2 \tag{104}$$

Solving Einstein's equations with this equation of state gives for the density:

$$\rho = \rho_0 F^{-3} \exp[1/2(1/F^2 - 1)] \tag{105}$$

where ρ_0 is the present density if we assume that the present value of F is 1, i.e. we assume that the value of c is equal to the value c_{eff} now. Any other choice would just shift the present value of F.

It can be seen that for very small values of F, $T_{\alpha\beta}$ viewed as an energy-momentum tensor having the form 103 corresponds to a scalar field:

$$\Phi = \frac{1}{F} \tag{106}$$

with potential:

$$V(\Phi) = -\frac{\rho_0}{6}\Phi^5 \exp[(1/2)(\Phi^2 - 1)] \tag{107}$$

On the other hand, assuming that F can reach sufficiently large values, the model behaves as if $T_{\alpha\beta}$ had the form 102 of a perfect fluid with:

$$p = 0 \tag{108}$$

i.e. as the energy-momentum of dust matter.

Although it is possible to conceive a model of the Universe for which the equation of state 104would hold during a small fraction of its history after the big-bang and recur to more common ideas afterwards, it is tempting to consider a model for which the equation of state would hold from the beginning to the end. This model depends then on two free parameters, the curvature constant k and the cosmological constant Λ. On the other hand to determine completely the model we need to know the present value of ρ_0. This is a crucial point, because the value to be assigned to this quantity depends on the interpretation that one wishes to give to the fluid described by the equation of state 104 at present time. Two possibilities can be considered: i) We can assume that this fluid is equivalent to the matter content of the universe, in which case the value of ρ_0 should be estimated as it is usually done: counting galaxies, guessing masses or else. This would be a complicated model not

very different from standard models and we do not consider it here anymore. ii) We can assume that ordinary matter is hierarchically organized with a fractal dimension less than 3 which means that it would be weightless compared to the energy of the background blackbody radiation at temperature $2.7\,K$. In other words matter would be just *The froth of the Universe*. This is a very simple and viable model to which Stefan's law assigns the following value to the present density:

$$\rho_0 = 4.5 \times 10^{-31}\,\mathrm{kg/m^3} \tag{109}$$

As usual two other quantities have to be derived from observation. Namely the Hubble constant H_0 and the deceleration parameter q_0. Considering as an example the following values:

$$H_0 = 75\,\mathrm{km/s/Mpc}, \quad q_0 = 0.1 \tag{110}$$

the model yields the following values for Λ and k:

$$\Lambda = -2. \times 10^{-51}\mathrm{m}^{-2}, \quad k = -7.3 \times 10^{-51}\mathrm{m}^{-2} \tag{111}$$

Appendix: Some definitions and notations

Given any space-time with line element:

$$ds^2 = g_{\alpha\beta}(x^\rho)dx^\alpha dx^\beta \quad \alpha, \beta, \dots = 0, 1, 2, 3 \quad x^0 = t, \tag{112}$$

let us consider a time-like congruence $\mathcal{R}$, u^α being its unit tangent vector field ($u_\alpha u^\alpha = -1$). The *Projector* into the plane orthogonal to u^α is:

$$\hat{g}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \tag{113}$$

By definition the *Newtonian field* is the opposite to the acceleration field:

$$\Lambda_\alpha = -u^\rho \nabla_\rho u_\alpha, \quad \Lambda_\alpha u^\alpha = 0. \tag{114}$$

The *Coriolis field*, or the rotation rate field, is the skew-symmetric 2-rank tensor orthogonal to u^α:

$$\Omega_{\alpha\beta} = \hat{\nabla}_\alpha u_\beta - \hat{\nabla}_\beta u_\alpha, \quad \Omega_{\alpha\beta} u^\alpha = 0. \tag{115}$$

where:

$$\hat{\nabla}_\alpha u_\beta \equiv \hat{g}^\rho_\alpha \hat{g}^\sigma_\beta \nabla_\rho u_\sigma. \tag{116}$$

And *Born's deformation rate field* is the symmetric 2-rank tensor orthogonal to u^α:

$$\Sigma_{\alpha\beta} = \hat{\nabla}_\alpha u_\beta + \hat{\nabla}_\beta u_\alpha, \quad \Sigma_{\alpha\beta} u^\alpha = 0, \tag{117}$$

Let x^α be a system of adapted coordinates, i.e., such that $u^i = 0$. We use the following notations:

$$\xi = \sqrt{-g_{00}}, \qquad \varphi_i = \xi^{-2} g_{0i}, \tag{118}$$

and:

$$\hat{g}_{ij} = g_{ij} + \xi^2 \varphi_i \varphi_j \quad . \tag{119}$$

which we shall call the *Fermat* quo-tensor of the congruence. Here and below quo-tensor refers to an object, well defined on the quotient manifold $\mathcal{V}_3 = \mathcal{V}_4/\mathcal{R}$, which covariant components are the space components of a tensor of $\mathcal{V}_4$ orthogonal to u^α.

The *Newtonian field* is then the quo-vector:

$$\Lambda_i = -(\hat{\partial}_i \ln \xi + \partial_t \varphi_i), \quad \hat{\partial}_i \cdot \equiv \partial_i \cdot + \varphi_i \partial_t \cdot \tag{120}$$

The *Coriolis field*, or Rotation rate field, is the skew-symmetric quo-tensor:

$$\Omega_{ij} = \xi(\hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i) \tag{121}$$

and the *Born's deformation rate field* is the symmetric quo-tensor:

$$\Sigma_{ij} = \hat{\partial}_t \hat{g}_{ij}, \quad \hat{\partial}_t = \xi^{-1} \partial_t \tag{122}$$

To these familiar geometrical objects it is necessary to add the following ones, [26], [27], [15]:

$$\hat{\Gamma}^i_{jk} = \frac{1}{2} \hat{g}^{is} (\tilde{\partial}_j \hat{g}_{ks} + \tilde{\partial}_k \hat{g}_{js} - \tilde{\partial}_s \hat{g}_{jk}), \tag{123}$$

which are the *Zel'manov-Cattaneo symbols*, and:

$$\hat{R}^i_{jkl} = \tilde{\partial}_k \hat{\Gamma}^i_{jl} - \tilde{\partial}_l \hat{\Gamma}^i_{jk} + \hat{\Gamma}^i_{sk}\hat{\Gamma}^s_{jl} - \hat{\Gamma}^i_{sl}\hat{\Gamma}^s_{jk} \tag{124}$$

which is the *Zel'manov-Cattaneo quo-tensor*,

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